

## The Spider's Spacewalk Derivation of $\sin'$ and $\cos'$

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*The College Mathematics Journal*, March 1995, Volume 26, Number 2, pp. 144-145.

The usual proofs of the derivatives of sine and cosine in introductory calculus involve limits. I shall outline a simple geometric derivation that avoids evaluating limits, based on the interpretation of the derivative as the instantaneous rate of change. The principle behind this proof is found in a late nineteenth-century calculus textbook by J. M. Rice and W. W. Johnson, *The Elements of the Differential Calculus, Founded on the Method of Rates or Fluxions* (Wiley, New York, 1874).

A spider walks with speed 1 in a circular path around the outside of a round satellite of radius 1, as shown in Figure 1. At time  $t$  the spider will have travelled a distance  $t$ , which corresponds to a central angle of  $t$  radians. The altitude of the spider, in the standard coordinate system, is  $y = \sin(t)$  and the spider is  $x = \cos(t)$  units to the right (or left) of the origin.

Now look how fast the spider is moving *upward*. Since the altitude of the spider at time  $t$  is  $y = \sin(t)$ , its upward velocity is  $y' = \sin'(t)$ . Oops!—The spider loses its footing at time  $t$ , and since the gravity in outer space is negligible, it continues with the same direction and speed. It moves a distance 1 in additional time  $\Delta t = 1$ , from point

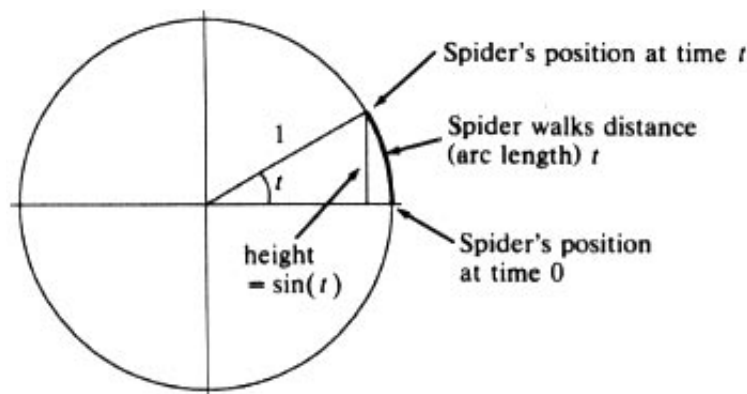


Figure 1

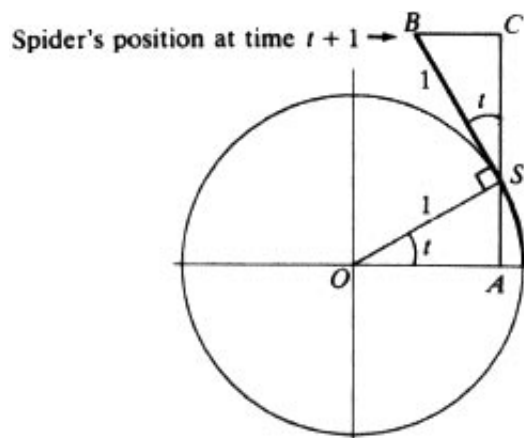


Figure 2

$S$  to point  $B$ ; see Figure 2. The spider's altitude changes by  $\Delta y = SC$ , so its upward velocity is  $y' = \Delta y / \Delta t = SC$ . But because triangles  $OAS$  and  $SCB$  are congruent ( $\angle OSA = 90^\circ - t$ , so  $\angle CSB = t = \angle AOS$ ),  $SC = OA = x = \cos(t)$ . Therefore  $\sin'(t) = y' = \cos(t)$ .

Similarly, the spider is  $x = \cos(t)$  units to the right of the center, and its horizontal velocity is  $x' = \cos'(t) = \Delta x / \Delta t = -BC = -SA = -\sin(t)$ . The minus sign arises because the spider is moving to the left when it is above the  $x$ -axis, i.e.,  $\Delta x / \Delta t$  is negative whenever  $y = \sin(t)$  is positive.

Figure 2 describes the case  $0 < t < \pi/2$ , but it is easily modified to yield the same results when the spider is located anywhere on the unit circle.

*Acknowledgement.* I thank Bob Gether and George Rosenstein for helpful comments and for the reference to Rice and Johnson's book.

[*Editor's note.* This proof also appeared in C. S. Ogilvy, Derivatives of  $\sin \theta$  and  $\cos \theta$ , *American Mathematical Monthly* 67 (1960) 673.]