N! and The Root Test

Charles C. Mumma II, University Of Washington, Seattle, WA 98195

American Mathematical Monthly, August–September 1986, Volume 93, Number 7, pg. 561.

I would be willing to bet that 99.98% of all freshman calculus students (perhaps even more!) attempting to determine the convergence of

$$\sum_{n=1}^{+\infty} \frac{2^n}{n!}$$

would *not* use the root test. This is not surprising, considering that most texts do not evaluate $\lim_{n\to+\infty} \sqrt[n]{n!}$. However, armed with the knowledge that this limit is $+\infty$, the root test becomes more versatile and accessible.

LEMMA. $\lim_{n\to+\infty} \sqrt[n]{n!} = +\infty.$

Proof. First notice that $(2n)! \geq \Pi_{k=n}^{2n} k \geq n^{n+1}$. Consequently

$$\sqrt[2n]{(2n)!} \geq \sqrt[2n]{n^{n+1}} \geq \sqrt{n} \quad \text{and} \quad \sqrt[2n+1]{(2n+1)!} \geq \sqrt[2n+1]{n^{n+1}} \geq \sqrt{n}.$$

A more accurate analysis of this limit yields a rather interesting result.

LEMMA. $\lim_{n\to+\infty} \dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}.$

Proof. Taking logarithms, we get

$$\ln\left(\frac{\sqrt[n]{n!}}{n}\right) = \frac{1}{n}\ln(n!) - \ln(n) = \frac{1}{n}\sum_{k=1}^{n}\ln(k) - \frac{1}{n}\sum_{k=1}^{n}\ln(n) = \sum_{k=1}^{n}\ln\left(\frac{k}{n}\right)\frac{1}{n}.$$

As $n \to +\infty$, this becomes $\displaystyle\int_0^1 \ln(x)\, dx = -1.$

This result is usually proved using the fact that the root test is stronger than the ratio test, that is,

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf \sqrt[n]{a_n} \leq \limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n},$$

where $(a_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of positive real numbers. The beauty of the present proof is its simplicity and directness and its use of methods readily available to freshman calculus students.