Elementary Proofs of Error Estimates for the Midpoint and Simpson's Rules

EDWARD C. FAZEKAS JR.
Department of Biostatistics
SUNY University at Buffalo
Buffalo, NY 14214
efazekas@buffalo.edu

PETER R. MERCER
Department of Mathematics
SUNY College at Buffalo
Buffalo, NY 14222
mercerpr@math.buffalostate.edu

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Usually encountered during a second course in calculus are the

$$\text{Trapezoid Rule,} \qquad \int_a^b f(x)\,dx \cong \frac{f(a)+f(b)}{2}(b-a),$$

and the

$$\text{Midpoint Rule,} \qquad \int_a^b f(x)\,dx \cong f\left(\frac{a+b}{2}\right)(b-a),$$

for a function f defined on $[a, b]$. The former rule approximates the integral by replacing the graph of f with the line segment from $(a, f(a))$ to $(b, f(b))$, while the latter approximates the integral by replacing the graph of f with the horizontal line segment through $(\frac{a+b}{2}, f(\frac{a+b}{2}))$.

In practice, the interval $[a, b]$ is divided up into a large number of subintervals and an approximation is applied over each subinterval. The hope is that the larger the number of subintervals, the better the approximation becomes. This hope, of course, rests on the nature of f itself—indeed f may be very complicated, or only partially known. This is why estimates for the accuracies of these approximations are important.

For example, if $|f''| \leq M$ on $[a, b]$ (at the endpoints, we mean derivatives from the right and from the left) then it is well known that for the

$$\text{Trapezoid Rule,} \qquad \left|\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a)\right| \leq M\frac{(b-a)^3}{12},$$

and for the

$$\text{Midpoint Rule,} \qquad \left|\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a)\right| \leq M\frac{(b-a)^3}{24}.$$

Typically these estimates are obtained via polynomial interpolation [**1**], which is by no means elementary. However, D. Cruz-Uribe and C. J. Neugebauer [**2**], obtained the Trapezoid Rule estimate above by a clever application of integration by parts, making it fully accessible to any calculus student.

Our investigation, which may serve as a companion to Cruz-Uribe and Neugebauer's paper, has two parts. First, we modify the idea used there to obtain the Midpoint Rule estimate. Then we extend the idea to obtain Simpson's Rule, thus addressing a problem posed in their paper namely, "it is an open problem to extend our ideas to give an elementary proof of this result."

The Trapezoid Rule Cruz-Uribe and Neugebauer employ what they call integration by parts "backwards." To illustrate what this means, we begin with an integrable function h on $[a, b]$, and we let H be an antiderivative:

$$H(t) = \int_a^t h(x)\,dx.$$

Then for another function f with bounded second derivative, integration by parts (twice) gives

$$\begin{aligned}\int_a^b f\,dh - hf\Big|_a^b &= -\int_a^b h\,df = -\int_a^b f'\,dH = \int_a^b H\,df' - f'H\Big|_a^b \\ &= \int_a^b H(t)f''(t)\,dt - f'H\Big|_a^b.\end{aligned}$$

That is,

$$\int_a^b f dh - hf\Big|_a^b = \int_a^b H(t)f''(t)\,dt - f'H\Big|_a^b. \tag{1}$$

Then in (1) Cruz-Uribe and Neugebauer essentially set

$$h(x) = x - \frac{a+b}{2},$$

in order to "pick up" the values of f at the endpoints a and b. See the graph of h in FIGURE 1 below.

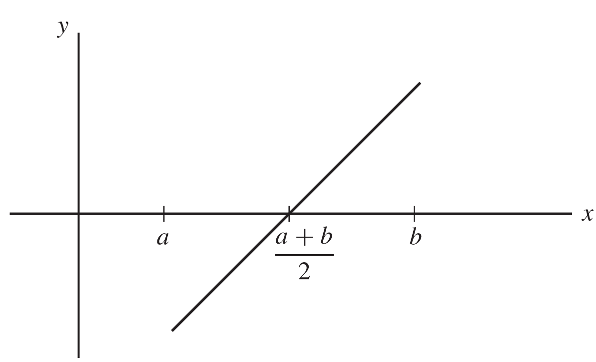

Figure 1 Graph of h for the Trapezoid Rule

Indeed, here $dh = dx$ and $H(a) = H(b) = 0$, so (1) becomes

$$\int_a^b f dh - hf\Big|_a^b = \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a) = \int_a^b H(t)f''(t)\,dt.$$

Now it is easily verified that $\int_a^b |H(t)|\,dt = \frac{(b-a)^3}{12}$, and so we get the Trapezoid Rule estimate

$$\left|\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a)\right| = \left|\int_a^b H(t)f''(t)\,dt\right|$$
$$\leq M\int_a^b |H(t)|\,dt = M\frac{(b-a)^3}{12}.$$

The Midpoint Rule Here instead, to try to "pick up" the values of f at the midpoint $\frac{a+b}{2}$, we'd like to set

$$h(x) = \begin{cases} x-a & \text{if } x \in [a, \frac{a+b}{2}] \\ x-b & \text{if } x \in (\frac{a+b}{2}, b], \end{cases}$$

but strictly speaking (1) does not apply, because h has a jump discontinuity at $(a+b)/2$. (FIGURE 2 shows the graph of h.) So we'll have to be a bit more careful, applying the formula to each half of the interval and taking limits.

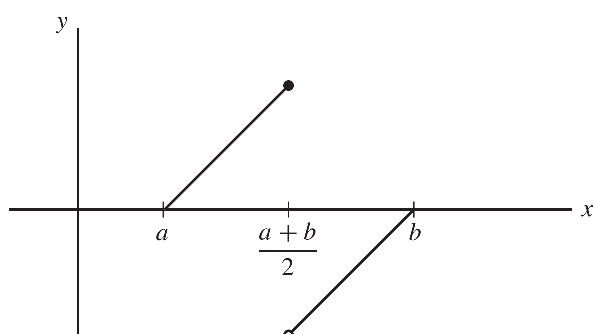

Figure 2 Graph of *h* for the Midpoint Rule

To allow for the jump at $c := (a+b)/2$, we modify (1) as follows:

$$\int_a^b f(x)\,dt - \lim_{\tau\to c^-} h(x)f(x)\Big|_a^\tau - \lim_{\tau\to c^+} h(x)f(x)\Big|_\tau^b$$
$$= \int_a^b H(t)f''(t)\,dt - f'H\Big|_a^b. \tag{2}$$

Still we have $dh = dx$ and $H(t) = \int_a^t h(x)\,dx$ satisfies $H(a) = H(b) = 0$, so (2) becomes

$$\int_a^b f(x)\,dt - \lim_{\tau\to c^-} h(x)f(x)\Big|_a^\tau - \lim_{\tau\to c^+} h(x)f(x)\Big|_\tau^b$$
$$= \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a) = \int_a^b H(t)f''(t)\,dt.$$

Thus, using $\int_a^b |H(t)|\,dt = \frac{(b-a)^3}{24}$, we obtain the Midpoint Rule estimate

$$\left|\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a)\right| = \left|\int_a^b H(t)f''(t)\,dt\right|$$
$$\leq M\int_a^b |H(t)|\,dt = M\frac{(b-a)^3}{24}.$$

More precise estimates As often happens in mathematics, if we ask for more then we get more. If we assume that f has a *continuous* second derivative, we can obtain more precise estimates, as follows. If we look carefully at the graph of the h used to obtain the Trapezoid Rule estimate, we can see that H does not change sign on $[a, b]$—it's nonpositive—and so by the Mean Value Theorem for Integrals [**1**] there exists $\xi_1 \in (a, b)$ such that

$$\int_a^b H(t)f''(t)\,dt = f''(\xi_1)\int_a^b H(t)\,dt.$$

As one may verify, $\int_a^b H(t)\,dt = -\frac{(b-a)^3}{12}$ and so we have for the

$$\text{Trapezoid Rule,}\qquad \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a) = -f''(\xi_1)\frac{(b-a)^3}{12}.$$

Likewise, the H from the Midpoint Rule estimate above does not change sign on $[a, b]$—it's nonnegative—and so there exists $\xi_2 \in (a, b)$ such that

$$\int_a^b H(t)f''(t)\,dt = f''(\xi_2)\int_a^b H(t)\,dt.$$

Here, $\int_a^b H(t)\,dt = \frac{(b-a)^3}{24}$, and so for the

$$\text{Midpoint Rule,}\qquad \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a) = f''(\xi_2)\frac{(b-a)^3}{24}\ .$$

Simpson's Rule Looking at the more precise error terms for the Trapezoid and Midpoint Rules, we notice that $\frac{1}{3}(-\frac{(b-a)^3}{12}) + \frac{2}{3}\frac{(b-a)^3}{24} = 0$. This suggests (as observed by many authors) that a quadrature rule

$$\frac{1}{3}(\text{Trapezoid Rule}) + \frac{2}{3}(\text{Midpoint Rule})$$

may be quite good. Indeed it is—this is Simpson's Rule.

Pursuing this idea according to what we have done above, we let

$$h_1(x) = x - \frac{a+b}{2} \qquad \text{and} \qquad h_2(x) = \begin{cases} x-a & \text{if } t \in [a, \frac{a+b}{2}] \\ x-b & \text{if } t \in (\frac{a+b}{2}, b]. \end{cases}$$

Now we apply (2) to $h = \frac{1}{3}h_1 + \frac{2}{3}h_2$ and $H(t) = \int_a^t h(x)\,dx$. We omit many of the details; they are elementary but admittedly tedious. The left side of (2) simplifies to become

$$\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

On the right side, one can check that $H(a) = H(b) = 0$, so we apply integration by parts as before to obtain

$$\int_a^b H(t)f''(t)\,dt - f'H\Big|_a^b = \int_a^b H(t)f''(t)\,dt$$

$$= H_1(t)f''(t)\Big|_a^b - \int_a^b H_1(t)f'''(t)\,dt,$$

where $H_1(t) = \int_a^t H(x)\,dx$. Again, $H_1(a) = H_1(b) = 0$, so we apply integration by parts another time to obtain

$$H_1(t)f''(t)\Big|_a^b - \int_a^b H_1(t)f'''(t)\,dt = -\int_a^b H_1(t)f'''(t)\,dt$$

$$= -H_2(t)f'''(t)\Big|_a^b + \int_a^b H_2(t)f^{(iv)}(t)\,dt,$$

where $H_2(t) = \int_a^t H_1(x)\,dx$. Here, yet again, $H_2(a) = H_2(b) = 0$, and so we have thus far

$$\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = \int_a^b H_2(t)f^{(iv)}(t)\,dt.$$

As before, $H_2(t)$ does not change sign on $[a, b]$ and so, for $f^{(iv)}$ continuous, by the Mean Value Theorem for Integrals there exists $\xi \in (a, b)$ such that

$$\int_a^b H_2(t)f^{(iv)}(t)\,dt = f^{(iv)}(\xi)\int_a^b H_2(t)\,dt.$$

Finally, evaluating $\int_a^b H_2(t)\,dt$, we obtain the classical error estimate [**1**] for

$$\text{Simpson's Rule,}\quad \int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= -f^{(iv)}(\xi)\frac{(b-a)^5}{2880}.$$

As mentioned, we omitted many of the details. But the intrepid reader may check, for example, that for $[a, b] = [0, 1]$ we have

$$0 \geq H_2(t) = \begin{cases} \frac{1}{72}(3t^4 - 2t^3) & t \in [0, 1/2] \\ \frac{1}{72}(3t^4 - 10t^3 + 12t^2 - 6t + 1) & t \in (1/2, 1]\,. \end{cases}$$

Simpson's Rule approximates the integral by replacing the graph of f with the parabola through $(a, f(a))$, $\left(\frac{a+b}{2}, f(\frac{a+b}{2})\right)$, and $(b, f(b))$. *Higher order quadrature rules* replace f with higher order polynomials. In principle, the above idea could be extended to these rules, but unless a manageable pattern can be discerned the computations quickly become unwieldy. As such, methods for obtaining estimates for higher order quadrature rules remain, for now, within the realm polynomial interpolation.

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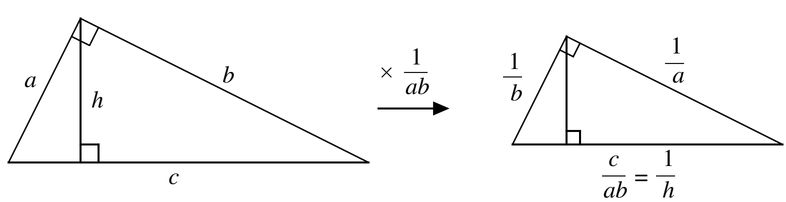
Proof Without Words: A Reciprocal Pythagorean Theorem

doi:10.4169/002557009X478427

If a and b are the legs and h the altitude to the hypotenuse c of a right triangle, then

$$\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 = \left(\frac{1}{h}\right)^2.$$

Proof.



NOTE: For another proof, see V. Ferlini, Mathematics without (many) words, *College Math. J.* **33** (2002) 170.

—Roger B. Nelsen
Lewis & Clark College
Portland, OR 97219

Geometry in Government

In a family of accomplished scholars, "my performance was decidedly mediocre. I approached the bulk of my schoolwork as a chore rather than an intellectual adventure."

Geometry came to the rescue. "Instead of memorizing facts, we were asked to think in clear, logical steps. Beginning from a few intuitive postulates, far-reaching consequences could be derived, and I took immediately to the sport of proving theorems."

—Steven Chu
United States Secretary of Energy
as quoted by Lisa M. Krieger
Bay Area News Group